THE EARLY UNIVERSE¹

Andrew R. Liddle
Astronomy Centre, University of Sussex, Falmer, Brighton BN1 9QH, U. K.

Abstract

An introductory account is given of the modern understanding of the physics of the early Universe. Particular emphasis is placed on the paradigm of *cosmological inflation*, which postulates a period of accelerated expansion during the Universe's earliest stages. Inflation provides a possible origin for structure in the Universe, such as microwave background anisotropies, galaxies and galaxy clusters; these observed structures can therefore be used to test models of inflation. A brief account is given of other early Universe topics, namely baryogenesis, topological defects, dark matter candidates and primordial black holes.

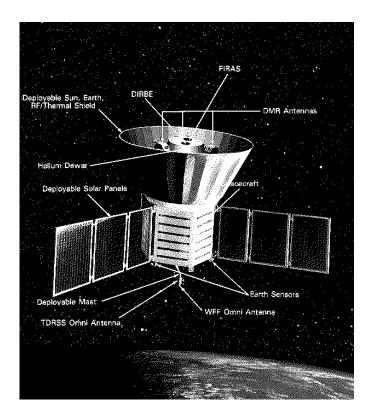


Figure 1: By providing the first measurement of irregularities in the microwave background radiation in 1992, the COBE satellite (artist's impression, courtesy NASA [reduced resolution in archive version]) revolutionized modern cosmology.

¹Preprint SUSSEX-AST 96/12-1, astro-ph/9612093 Lectures given at the Winter School 'From Quantum Fluctuations to Cosmological Structures' in Casablanca, Morocco, December 1996. Please notify any errors to a.liddle@sussex.ac.uk

Contents

1	Intr	roduction	3
2	$\mathbf{A}\mathbf{n}$	Early Universe Overview	3
3	A B	Big Bang Reminder	4
	3.1	Equations of motion	5
	3.2	Characteristic scales	6
4	Dno	blems with the Big Bang	7
4	4.1	The flatness problem	7
	4.1	The horizon problem	7
	4.3	The monopole problem (and other relics)	7
	$\frac{4.3}{4.4}$	Homogeneity and isotropy	8
	4.4	nomogeneity and isotropy	C
5	The	e Idea of Inflation	8
	5.1	The flatness problem	9
	5.2	Relic abundances	9
	5.3	The horizon problem and homogeneity	10
6	Mod	delling the Inflationary Expansion	10
U	6.1	Scalar fields and their potentials	10
	6.2	Equations of motion and solutions	12
	6.3	A worked example: polynomial chaotic inflation	14
	6.4	The relation between inflation and slow-roll	14
	6.5	Reheating after inflation	15
	6.6	The range of inflation models	15
	0.0	6.6.1 Chaotic inflation models	16
		6.6.2 Multi-field theories	16
		6.6.3 Beyond general relativity	17
		6.6.4 Open inflation	18
	6.7	Recap	18
	0.1	itecap	10
7	Den	sity Perturbations and Gravitational Waves	18
	7.1	Production during inflation	19
		A worked example	22
	7.3	Observational consequences	22
	7.4	Tests of inflation	23
8	Fur	ther Early Universe Topics	24
	8.1	Baryogenesis	24
	8.2	Topological defects	26
	8.3	Dark matter	28
		8.3.1 Neutrinos	28
		8.3.2 WIMPS and axions	29
	8.4	Primordial black holes	29
_	~		
9	Sun	amary	30

1 Introduction

The early universe has been a major topic of research now for more than twenty years, and has come to encompass a wide range of different topics. The common theme is the introduction of ideas from particle physics, such as symmetry breaking and the unification of fundamental forces, into a cosmological setting. Conventionally, the 'Early Universe' refers to those epochs during which the Universe was so hot and energetic that the appropriate physics is at an energy scale inaccessible even to the largest particle accelerators. The key tool therefore is speculative extrapolation of known physics into the realm of the unknown. Despite this, however, many of the key ideas can be understood without requiring any deep knowledge of particle physics, and in these lectures I shall aim to discuss the very early Universe from as astronomical a viewpoint as possible.

Although as a subject the early Universe has become quite a mature one, some of the most spectacular developments have been extremely recent, within the last few years. This is because for the first time it is becoming possible to probe the physics of the early Universe observationally, through the increasing range of observations of structure in the Universe. One of the most important realizations in this subject was that the standard cosmology, based on known physics, is incapable of providing a theory for the origin of structure in the Universe. Instead, any theory for that origin must lie in the early Universe. This has become the most important motivation for studying this subject, and has the important consequence that observations of structure in the Universe can be used to constrain models of the early Universe. Indeed, for the first time many proposals for the physics of the early Universe have been ruled out, a sure sign that the area is becoming a proper, falsifiable, science. The pivotal point in these new developments was the discovery in 1992 of anisotropies in the microwave background radiation, by the COsmic Background Explorer (COBE) satellite (shown in Figure 1) [1, 2].

A central element of early Universe cosmology is the paradigm of cosmological inflation. This proposes a period of accelerated expansion in the Universe's distant past. While introduced to solve a set of largely conceptual problems concerning the initial conditions for the big bang theory, it was rapidly realized that it also provides a theory for the origin of structure in the Universe. I shall devote most of my time to inflation, especially since it, amongst all early Universe topics, is of the most direct relevance to the other lectures at this School. Only near the end will I make some discussion of some other research areas which lie within the early Universe heading.

The rapid recent developments in this area have rendered some of the literature rather obsolete in parts, but there are several good references. The classic early Universe textbook is the one of that name, by Kolb & Turner [3]. This is the only textbook to cover the entire early Universe subject area. The book by Linde [4] concentrates on inflation, with a strong emphasis on particle physics aspects. Studies of structure in the Universe have been the topic of several recent books [5]. A very nice review of inflation, written specifically with astronomers in mind but unfortunately pre-dating COBE, is that of Narlikar & Padmanabhan [6], and a more recent review of the relation between inflationary cosmology and structure in the universe was given by myself and Lyth [7].

2 An Early Universe Overview

The standard hot big bang theory is an extremely successful one, passing some crucial observational tests, of which I'd highlight five.

- The expansion of the Universe.
- The existence and spectrum of the cosmic microwave background radiation.
- The abundances of light elements in the Universe (nucleosynthesis).
- That the predicted age of the Universe is comparable to direct age measurements of objects within the Universe.
- That given the irregularities seen in the microwave background by COBE, there exists a reasonable explanation for the development of structure in the Universe, through gravitational collapse.

In combination, these are extremely compelling. However, the standard hot big bang theory is limited to those epochs where the Universe is cool enough that the underlying physical processes are well established and understood through terrestrial experiment. It does not attempt to address the state of the Universe at earlier, hotter, times. Furthermore, the hot big bang theory leaves a range of crucial questions unanswered, for it turns out that it can successfully proceed only if the initial conditions are very carefully chosen. The assumption of early Universe studies is that the mysteries of the conditions under which the big bang theory operates may be explained through the physics occurring in its distant, unexplored past. If so, accurate observations of the present state of the Universe may highlight the types of process occurring during these early stages, and perhaps even shed light on the nature of physical laws at energies which it would be inconceivable to explore by other means.

The types of question that Early Universe Cosmology strives to answer are the following.

- What governs the global structure of the Universe?
 - Why is the large-scale Universe so close to spatial flatness?
 - Why is the matter in the Universe so homogeneously (ie evenly) distributed on large scales?
- What is the origin of structure in the universe (microwave background anisotropies, galaxies, galaxy clusters, etc)?
- Why is there far more matter than antimatter in the universe?
- What is the nature of the matter in the Universe? Is there any dark matter, and if so how much and what are its properties?
- What are the consequences of exotic particle theories at high energies?
 - Are topological defects (domain walls, cosmic strings, monopoles) produced in the early universe?
 - Are primordial black holes produced in the early Universe?
 - Do unusual particles such as axions exist?

3 A Big Bang Reminder

In this Section I'll provide a brief reminder of the standard hot big bang theory, and establish the notation I'll use throughout.

3.1Equations of motion

The hot big bang theory is based on the cosmological principle, which states that the Universe should look the same to all observers. That tells us that the universe must be homogeneous and isotropic, which in turn tells us which metric must be used to describe it. It is the Robertson-Walker metric

$$ds^{2} = -dt^{2} + a^{2}(t) \left[\frac{dr^{2}}{1 - kr^{2}} + r^{2} \left(d\theta^{2} + \sin^{2}\theta \, d\phi^{2} \right) \right]. \tag{1}$$

Here t is the time variable, and $r-\theta-\phi$ are (polar) coordinates. The constant k measures the spatial curvature, with k negative, zero and positive corresponding to open, flat and closed universes respectively. If k is zero or negative, then the range of r is from zero to infinity and the universe is infinite, while if k is positive then r goes from zero to $1/\sqrt{k}$. Usually the coordinates are rescaled to make k equal to -1, 0 or +1. The quantity a(t) is the scale-factor of the Universe, which measures how rapidly it is expanding. The form of a(t) depends on the type of material within the Universe, as we'll see.

If no external forces are acting, then a particle at rest at a given set of coordinates (r, θ, ϕ) will remain there. Such coordinates are said to be comoving with the expansion. One swaps between physical (ie actual) and comoving distances via

physical distance =
$$a(t) \times \text{comoving distance}$$
. (2)

The expansion of the Universe is governed by the properties of material within it. This can be specified by the energy density $\rho(t)$ and the pressure p(t). These are often related by an equation of state, which gives p as a function of ρ ; the classic examples are

$$p = \frac{\rho}{3}$$
 Radiation, (3)
 $p = 0$ Non-relativistic matter. (4)

$$p = 0$$
 Non-relativistic matter. (4)

In general though there need not be a simple equation of state; for example there may be more than one type of material, such as a combination of radiation and non-relativistic matter, and certain types of material, such as a scalar field (a type of material crucial for modelling inflation), cannot be described by an equation of state at all.

The crucial equations describing the expansion of the Universe are

$$H^2 = \frac{8\pi}{3m_{\rm Pl}^2} \rho - \frac{k}{a^2}$$
 Friedmann equation (5)

$$\dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} = 0$$
 Fluid equation (6)

where overdots are time derivatives and $H = \dot{a}/a$ is the Hubble parameter. The terms in the fluid equation contributing to $\dot{\rho}$ have a simple interpretation; the term $3H\rho$ is the reduction in density due to the increase in volume, and the term 3Hp is the reduction in energy caused by the thermodynamic work done by the pressure when this expansion occurs.

 $^{^2}$ I follow standard cosmological practice of setting the fundamental constants c and \hbar equal to one. This makes the energy density and mass density interchangeable (since the former is c^2 times the latter). I shall also normally use the Planck mass $m_{\rm Pl}$ rather than the gravitational constant G; with the convention just mentioned they are related by $G \equiv m_{\rm Pl}^{-2}$.

These can also be combined to form a new equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3m_{\rm Pl}^2} \left(\rho + 3p\right) \qquad \text{Acceleration equation} \tag{7}$$

in which k does not appear explicitly.

The spatial geometry is flat if k = 0. For a given H, this requires that the density equals the critical density

$$\rho_{\rm c}(t) = \frac{3m_{\rm Pl}^2 H^2}{8\pi} \,. \tag{8}$$

Densities are often measured as fractions of ρ_c :

$$\Omega(t) \equiv \frac{\rho}{\rho_c} \,. \tag{9}$$

The present value of the Hubble parameter is still not that well known, and is normally parametrized as

$$H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1} = \frac{h}{3000} \text{ Mpc}^{-1},$$
 (10)

where h is normally assumed to lie in the range $0.5 \le h \le 0.8$. The present critical density is

$$\rho_{\rm c}(t_0) = 1.88 \,h^2 \times 10^{-29} \,\mathrm{g \, cm^{-3}} = 2.77 \,h^{-1} \times 10^{11} \,M_{\odot} / (h^{-1} \mathrm{Mpc})^3 \,. \tag{11}$$

The simplest solutions to these equations arise when a simple equation of state is chosen

Matter Domination
$$p = 0$$
: $\rho \propto a^{-3}$ $a(t) \propto t^{2/3}$ (12)

Radiation Domination
$$p = \rho/3$$
: $\rho \propto a^{-4}$ $a(t) \propto t^{1/2}$ (13)

Cosmological Constant
$$p = -\rho$$
: $\rho = \text{constant}$ $a(t) \propto \exp(Ht)$ (14)

3.2 Characteristic scales

The big bang universe has two characteristic scales

- The Hubble time/length H^{-1} .
- The curvature scale $a|k|^{-1/2}$.

The first of these gives the characteristic timescale of evolution of a(t), and the second gives the distance up to which space can be taken as flat. As written above they are both physical scales; to obtain the corresponding comoving scale one should divide by a(t). The ratio of these scales actually gives a measure of Ω ; from the Friedmann equation we find

$$\sqrt{|\Omega - 1|} = \frac{H^{-1}}{a|k|^{-1/2}}. (15)$$

A crucial property of the big bang universe is that it possesses *horizons*; even light can only travel a finite distance since the start of the universe t_* , given by

$$d_{\rm H}(t) = a(t) \int_{t_*}^{t} \frac{dt}{a(t)} \,.$$
 (16)

For example, matter domination gives $d_{\rm H}(t) = 3t = 2H^{-1}$. In a big bang universe, $d_{\rm H}(t_0)$ is a good approximation to the distance to the surface of last scattering, since $t_0 \gg t_{\rm decoupling}$.

Problems with the Big Bang 4

In this Section I shall quickly review the original motivation for the inflationary cosmology. These problems were largely one of initial conditions. While historically these problems were very important, they are now somewhat marginalized as focus is instead concentrated on inflation as a theory for the origin of cosmic structure.

4.1 The flatness problem

The Friedmann equation can be written in the form

$$|\Omega - 1| = \frac{|k|}{a^2 H^2} \,. \tag{17}$$

During standard big bang evolution, a^2H^2 is decreasing, and so Ω moves away from one, eg

Matter domination:
$$|\Omega - 1| \propto t^{2/3}$$
 (18)

Radiation domination:
$$|\Omega - 1| \propto t$$
 (19)

where the solutions apply provided Ω is close to one. So $\Omega = 1$ is an unstable critical point. Since we know that today Ω is certainly within an order of magnitude of one, it must have been much closer in the past, eg

nucleosynthesis
$$(t \sim 1 \text{ sec})$$
: $|\Omega - 1| < \mathcal{O}(10^{-16})$ (20)

nucleosynthesis
$$(t \sim 1 \text{ sec})$$
: $|\Omega - 1| < \mathcal{O}(10^{-16})$ (20)
electro-weak scale $(t \sim 10^{-11} \text{ sec})$: $|\Omega - 1| < \mathcal{O}(10^{-27})$ (21)

That is, hardly any choices of the initial density lead to a Universe like our own. Typically, the Universe will either swiftly recollapse, or will rapidly expand and cool below 3K within its first second of existence.

4.2 The horizon problem

Microwave photons emitted from opposite sides of the sky appear to be in thermal equilibrium at almost the same temperature. Yet there was no time for those regions to interact before the photons were emitted, because of the finite horizon size.

$$\int_{t_*}^{t_{\text{dec}}} \frac{dt}{a(t)} \ll \int_{t_{\text{dec}}}^{t_0} \frac{dt}{a(t)}. \tag{22}$$

In fact, any regions separated by more than about 2 degrees would be causally separated at decoupling in the hot big bang theory.

The monopole problem (and other relics) 4.3

Modern particle theories predict a variety of 'unwanted relics', which would violate observations. These include

- Magnetic monopoles.
- Domain walls.
- Supersymmetric particles.

• 'Moduli' fields associated with superstrings.

Typically, the problem is that these are expected to be created very early in the Universe's history, during the radiation era. But because they are diluted by the expansion more slowly than radiation (eg a^{-3} instead of a^{-4}) it is very easy for them to become the dominant material in the universe, in contradiction to observations. One has to dispose of them without harming the conventional matter in the universe.

Homogeneity and isotropy 4.4

This discussion is a variant on the horizon problem discussion given previously. The COBE satellite sees irregularities on all accessible angular scales, from a few degrees upwards. In the simplest cosmological models, where these irregularities are intrinsic to the last scattering surface, the perturbations are on too large a scale to have been created between the big bang and the time of decoupling, because the horizon size at decoupling subtends only a degree or so. Hence these perturbations must have been part of the initial conditions.³

If this is the case, then the hot big bang theory does not allow a predictive theory for the origin of structure. While there is no reason why it is required to give a predictive theory, this would be a major setback and disappointment for the study of structure formation in the Universe.

5 The Idea of Inflation

Seen with many years of hindsight, the idea of inflation is actually rather obvious. Take for example the Friedmann equation as used to analyze the flatness problem

$$|\Omega - 1| = \frac{|k|}{a^2 H^2} \,. \tag{23}$$

The problem with the hot big bang model is that aH always decreases, and so Ω is repelled

Reverse this! Define inflation to be any epoch where $\ddot{a} > 0$, an accelerated expansion. We can rewrite this in several different ways

INFLATION
$$\iff \ddot{a} > 0$$
 (24)

$$\iff \frac{d(H^{-1}/a)}{dt} < 0 \tag{25}$$

$$\iff p < -\frac{\rho}{3} \tag{26}$$

$$\iff p < -\frac{\rho}{3}$$
 (26)

The middle definition is my favourite, because it has the most direct geometrical interpretation. It says that the Hubble length, as measured in comoving coordinates, decreases during inflation. At any other time, the comoving Hubble length increases. This is the key property of inflation; although typically the expansion of the Universe is very rapid, the crucial characteristic scale of the Universe is actually becoming smaller, when measured relative to that expansion.

Since the successes of the hot big bang theory rely on the Universe having a conventional (non-inflationary) evolution, we cannot permit this inflationary period to go on forever — it

³Note though that it is not yet known for definite that there are large-angle perturbations intrinsic to the last scattering surface. For example, in a topological defect model such as cosmic strings, such perturbations could be generated as the microwave photons propagate towards us.

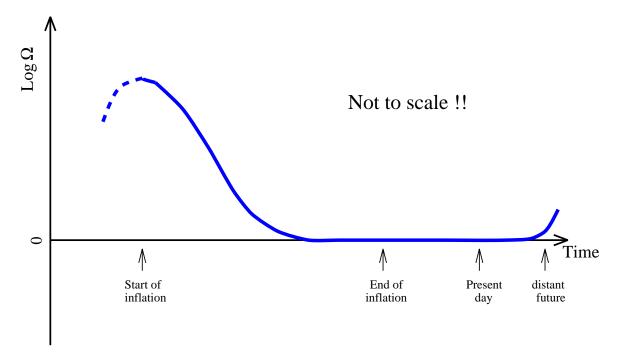


Figure 2: A possible evolution of Ω . There may or may not be evolution before inflation, shown by the dotted line. During inflation Ω is forced dramatically towards one, and remains there right up to the present. Only in the extremely distant future will it begin to evolve away from one again.

must come to an end early enough that the big bang successes are not threatened. Normally, then, inflation is viewed as a phenomenon of the very early universe, which comes to an end and is followed by the conventional behaviour. Inflation does not replace the hot big bang theory; it is a bolt-on accessory attached at early times to improve the performance of the theory.

5.1 The flatness problem

We can now, more or less by definition, solve the flatness problem. From its definition (eg the middle condition above), inflation is precisely the condition that Ω is forced towards one rather than away from it. As we shall see, this typically happens very rapidly. All we need is to have enough inflation that Ω is moved so close to one during the inflationary epoch that it stays very close to one right to the present, despite being repelled from one for all the post-inflationary period. Obtaining sufficient inflation to perform this task is actually fairly easy. A schematic illustration of this behaviour is shown in Figure 2.

5.2 Relic abundances

The rapid expansion of the inflationary stage rapidly dilutes the unwanted relic particles, because the energy density during inflation falls off more slowly (as a^{-2} or slower) than the relic particle density. Very quickly their density becomes negligible.

This resolution can only work if, after inflation, the energy density of the Universe can be turned into conventional matter without recreating the unwanted relics. This can be achieved by ensuring that during the conversion, known as *reheating*, the temperature never gets hot

enough again to allow their thermal recreation. Then reheating can generate solely the things which we want. Such successful reheating allows us to get back into the hot big bang universe, recovering all its later successes such as nucleosynthesis and the microwave background.

5.3 The horizon problem and homogeneity

The inflationary expansion also solves the horizon problem. The basic strategy is to ensure that

$$\int_{t_*}^{t_{\text{dec}}} \frac{dt}{a(t)} \gg \int_{t_{\text{dec}}}^{t_0} \frac{dt}{a(t)}, \tag{27}$$

so that light can travel much further before decoupling than it can afterwards. This cannot be done with standard evolution, but can be achieved by inflation.

An alternative way to view this is to remember that inflation corresponds to a decreasing comoving Hubble length. The Hubble length is ordinarily a good measure of how far things can travel in the universe; what this is telling us is that the region of the Universe we can see after (even long after) inflation is much smaller than the region which would have been visible before inflation started. Hence causal physics was perfectly capable of producing a large smooth thermalized region, encompassing a volume greatly in excess of our presently observable universe. In Figure 3, the outer circle indicates the initial Hubble length, encompassing the shaded smooth patch. Inflation shrinks this dramatically inwards towards the dot indicating our position, and then after inflation it increases while staying within the initial smooth patch.

Equally, causal processes would be capable of generating irregularities in the Universe on scales greatly exceeding our presently observable universe. I'll have much more to say about that soon.

6 Modelling the Inflationary Expansion

We have seen that a period of accelerated expansion — inflation — is sufficient to resolve a range of cosmological problems. But we need a plausible scenario for driving such an expansion if we are to be able to make proper calculations. This is provided by cosmological scalar fields.

6.1 Scalar fields and their potentials

In particle physics, a scalar field is used to represent spin zero particles. It transforms as a scalar (that is, it is unchanged) under coordinate transformations. In a homogeneous universe, the scalar field is a function of time alone.

In particle theories, scalar fields are a crucial ingredient for spontaneous symmetry breaking. The most famous example is the Higgs field which breaks the electro-weak symmetry, whose existence is hoped to be verified at the Large Hadron Collider at CERN when it commences experiments next millenium. Scalar fields are also expected to be associated with the breaking of other symmetries, such as those of Grand Unified Theories, supersymmetry etc.

⁴Although this is a standard description, it isn't totally accurate. A more accurate argument is as follows [7]. At the beginning of inflation particles are distributed in a set of modes. This may be a thermal distribution or something else; whatever, since the energy density is finite there will be a shortest wavelength occupied mode, e.g. for a thermal distribution $\lambda_{\text{max}} \sim 1/T$. Expressed in physical coordinates, once inflation has stretched all modes including this one to be much larger than the Hubble length, the Universe becomes homogeneous. In comoving coordinates, the equivalent picture is that the Hubble length shrinks in until it's much smaller than the shortest wavelength, and the universe, as before, appears homogeneous.

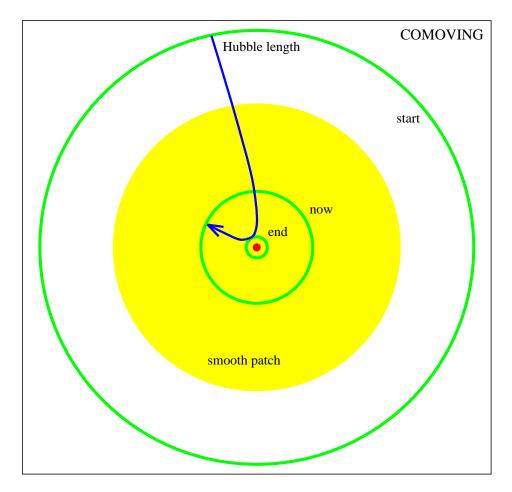


Figure 3: Solving the horizon problem. Initially the Hubble length is large, and a smooth patch forms by causal interactions. Inflation then shrinks the Hubble length, and even the subsequent expansion again after inflation leaves the observable universe within the smoothed patch.

- Any specific particle theory (eg GUTS, superstrings) contains scalar fields.
- No fundamental scalar field has yet been observed.
- In condensed matter systems (such as superconductors, superfluid helium etc) scalar fields are widely observed, associated with any phase transition. People working in that subject normally refer to the scalar fields as 'order parameters'.

The starting point I'll use for our investigation is the expressions for the effective energy density and pressure of a homogeneous scalar field, which I'll call ϕ . They are

$$\rho_{\phi} = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

$$p_{\phi} = \frac{1}{2}\dot{\phi}^2 - V(\phi).$$
(28)

$$p_{\phi} = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{29}$$

One can think of the first term in each as a kinetic energy, and the second as a potential energy. The potential energy $V(\phi)$ can be thought of as a form of 'configurational' or 'binding' energy;

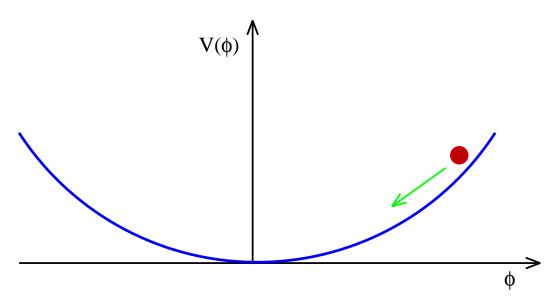


Figure 4: A generic inflationary potential.

it measures how much internal energy is associated with a particular field value. Normally, like all systems, scalar fields try to minimize this energy; however, a crucial ingredient which allows inflation is that scalar fields are not always very efficient at reaching this minimum energy state.

Note in passing that a scalar field cannot in general be described by an equation of state; there is no unique value of p that can be associated with a given ρ as the energy density can be divided between potential and kinetic energy in different ways.

In a given theory, there would be a specific form for the potential $V(\phi)$, at least up to some parameters which one could hope to measure (such as the effective mass and interaction strength of the scalar field). However, we are not presently in a position where there is a well established fundamental theory that one can use, so, in the absence of such a theory, inflation workers tend to regard $V(\phi)$ as a function to be chosen arbitrarily, with different choices corresponding to different models of inflation (of which there are many). Some example potentials are

$$V(\phi) = \lambda \left(\phi^2 - M^2\right)^2$$
 Higgs potential (30)
 $V(\phi) = \frac{1}{2}m^2\phi^2$ Massive scalar field (31)
 $V(\phi) = \lambda\phi^4$ Self-interacting scalar field (32)

$$V(\phi) = \frac{1}{2}m^2\phi^2$$
 Massive scalar field (31)

$$V(\phi) = \lambda \phi^4$$
 Self-interacting scalar field (32)

The strength of this approach is that it seems possible to capture many of the crucial properties of inflation by looking at some simple potentials; one is looking for results which will still hold when more 'realistic' potentials are chosen. Figure 4 shows such a generic potential, with the scalar field displaced from the minimum and trying to reach it.

6.2Equations of motion and solutions

The equations for an expanding universe containing a homogeneous scalar field are easily obtained by substituting Eqs. (28) and (29) into the Friedmann and fluid equations, giving

$$H^2 = \frac{8\pi}{3m_{\rm Pl}^2} \left[V(\phi) + \frac{1}{2}\dot{\phi}^2 \right] , \qquad (33)$$

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi), \qquad (34)$$

where prime indicates $d/d\phi$.

Since

$$\ddot{a} > 0 \iff p < -\frac{\rho}{3}$$
 $\iff \dot{\phi}^2 < V(\phi)$ (35)

we will have inflation whenever the potential energy dominates. This should be possible provided the potential is flat enough, as the scalar field would then be expected to roll slowly. The potential should also have a minimum in which inflation can end.

The standard strategy for solving these equations is the **slow-roll approximation** (SRA); this assumes that a term can be neglected in each of the equations of motion to leave the simpler set

$$H^2 \simeq \frac{8\pi}{3m_{\rm Pl}^2}V \tag{36}$$

$$3H\dot{\phi} \simeq -V' \tag{37}$$

If we define slow-roll parameters [8]

$$\epsilon(\phi) = \frac{m_{\rm Pl}^2}{16\pi} \left(\frac{V'}{V}\right)^2 \quad ; \quad \eta(\phi) = \frac{m_{\rm Pl}^2}{8\pi} \frac{V''}{V},$$
(38)

where the first measures the slope of the potential and the second the curvature, then necessary conditions for the slow-roll approximation to hold are⁵

$$\epsilon \ll 1 \quad ; \quad |\eta| \ll 1 \,. \tag{39}$$

Unfortunately, although these are necessary conditions for the slow-roll approximation to hold, they are not sufficient.

- The SRA reduces the order of the system of equations by one, and so its general solution contains one less initial condition. It works only because one can prove [9, 10] that the solution to the full equations possesses an attractor property, eliminating the dependence on the extra parameter.
- A more elaborate version of the SRA exists which is sufficient as well as necessary [10].
- In Section 6.4 I'll show, with some caveats, that if the slow-roll approximation is valid then one has inflation.

The amount of inflation is normally specified by the logarithm of the amount of expansion, the number of e-foldings N given by

$$N \equiv \ln \frac{a(t_{\text{end}})}{a(t_{\text{initial}})} = \int_{t_{\text{i}}}^{t_{\text{e}}} H dt, \qquad (40)$$

$$\simeq -\frac{8\pi}{m_{\rm Pl}^2} \int_{\phi_{\rm i}}^{\phi_{\rm e}} \frac{V}{V'} d\phi, \qquad (41)$$

⁵Note that ϵ is positive by definition, whilst η can have either sign.

where the final step uses the SRA. Notice that the amount of inflation between two scalar field values can be calculated without needing to solve the equations of motion, and also that it is unchanged if one multiplies $V(\phi)$ by a constant.

The minimum amount of inflation required to solve the various cosmological problems is about 70 e-foldings, i.e. an expansion by a factor of 10^{30} . Although this looks large, inflation is typically so rapid that most inflation models give much more.

6.3 A worked example: polynomial chaotic inflation

The simplest inflation model [4] arises when one chooses a polynomial potential, such as that for a massive but otherwise non-interacting field, $V(\phi) = m^2 \phi^2/2$ where m is the mass of the scalar field. With this potential, the slow-roll equations are

$$3H\dot{\phi} + m^2\phi = 0 \quad ; \quad H^2 = \frac{4\pi m^2\phi^2}{3m_{\rm Pl}^2} \,,$$
 (42)

and the slow-roll parameters are

$$\epsilon = \eta = \frac{m_{\rm Pl}^2}{4\pi\phi^2} \,. \tag{43}$$

So inflation can proceed provided $|\phi| > m_{\rm Pl}/\sqrt{4\pi}$. The slow-roll solutions are

$$\phi(t) = \phi_{\rm i} - \frac{m \, m_{\rm Pl}}{\sqrt{12\pi}} t \,, \tag{44}$$

$$a(t) = a_{\rm i} \exp\left[\sqrt{\frac{4\pi}{3}} \frac{m}{m_{\rm Pl}} \left(\phi_{\rm i} t - \frac{m m_{\rm Pl}}{\sqrt{48\pi}} t^2\right)\right], \tag{45}$$

(where $\phi = \phi_i$ and $a = a_i$ at t = 0) and the total amount of inflation is

$$N_{\text{tot}} = 2\pi \frac{\phi_{\text{i}}^2}{m_{\text{Pl}}^2} - \frac{1}{2}.$$
 (46)

In order for classical physics to be valid we require $V \ll m_{\rm Pl}^4$, but it is still easy to get enough inflation provided m is small enough. As we shall later see, m is in fact required to be small from observational limits on the size of density perturbations produced.

6.4 The relation between inflation and slow-roll

The inflationary condition $\ddot{a} > 0$ is satisfied for a much wider range of behaviours than just (quasi-)exponential expansion. A classic example is power-law inflation $a \propto t^p$ for p > 1, which is an exact solution for an exponential potential

$$V(\phi) = V_0 \exp\left[-\sqrt{\frac{16\pi}{p}} \frac{\phi}{m_{\rm Pl}}\right]. \tag{47}$$

We can manipulate the condition for inflation as

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 > 0$$

$$\iff -\frac{\dot{H}}{H^2} < 1$$

$$\iff \frac{m_{\rm Pl}^2}{16\pi} \left(\frac{V'}{V}\right)^2 < 1$$

where the last manipulation uses the slow-roll approximation. The final condition is just the slow-roll condition $\epsilon < 1$, and hence

Slow-roll
$$\Longrightarrow$$
 Inflation

However, the converse is not strictly true, since we had to use the SRA in the derivation. However, in practice

$$\begin{array}{ccc} \text{Inflation} & \stackrel{\sim}{\Longrightarrow} & \epsilon < 1 \\ \text{Prolonged inflation} & \stackrel{\sim}{\Longrightarrow} & \eta < 1 \end{array}$$

6.5 Reheating after inflation

During inflation, all matter except the scalar field (usually called the inflaton) is redshifted to extremely low densities. **Reheating** is the process whereby the inflaton's energy density is converted back into conventional matter after inflation, re-entering the standard big bang theory.

Once the slow-roll conditions break down, the scalar field begins to move rapidly on the Hubble timescale, and oscillates at the bottom of the potential. As it does so, it decays into conventional matter. The details of reheating (as long as we believe that it occurs!) are not important for our considerations here. I'll just note that recently there has been quite a dramatic change of view as to how reheating takes place. Traditional treatments (e.g. Ref. [3]) added a phenomenological decay term; this was constrained to be very small and hence reheating was viewed as being very inefficient. This allowed substantial redshifting to take place after the end of inflation and before the universe returned to thermal equilibrium; hence the reheat temperature would be lower, by several orders of magnitude, than suggested by the energy density at the end of inflation.

This picture is radically revised in work by Kofman, Linde & Starobinsky [11] (see also Ref. [12]), who suggest that the decay can undergo broad parametric resonance, with extremely efficient transfer of energy from the coherent oscillations of the inflaton field. This initial transfer has been dubbed *preheating*. With such an efficient start to the reheating process, it now appears possible that the reheating epoch may be very short indeed and hence that most of the energy density in the inflaton field at the end of inflation may be available for conversion into thermalized form. The full consequences of this change in viewpoint remain to be fully investigated.

6.6 The range of inflation models

Over the last ten years or so a great number of inflationary models have been devised, both with and without reference to specific underlying particle theories. Here I will discuss a very small subset of the models which have been introduced.

However, as we shall be discussing in the next Section, observations have great prospects for distinguishing between the different inflationary models. By far the best type of observation for this purpose appears to be high resolution satellite microwave background anisotropy observations, and we are fortunate that two proposals have recently been approved — NASA has funded the MAP satellite [13] for launch around 2000, and ESA has approved the hopefully soon to be renamed COBRAS/SAMBA satellite [14] for launch a few years later. These satellites should offer very strong discrimination between the inflation models I shall now discuss. Indeed, it may even be possible to attempt a more challenging type of observation — one

which is independent of the particular inflationary model and hence begins to test the idea of inflation itself.

6.6.1 Chaotic inflation models

This is the standard type of inflation model [4]. The ingredients are

- A single scalar field, rolling in ...
- A potential $V(\phi)$, which in some regions satisfies the slow-roll conditions, while also possessing a minimum with zero potential in which inflation is to end.
- Initial conditions well up the potential, due to large fluctuations at the Planck era.

There are a large number of models of this type. Some are

Polynomial chaotic inflation $V(\phi) = \frac{1}{2}m^2\phi^2$ $V(\phi) = \lambda\phi^4$ Power-law inflation $V(\phi) = V_0 \exp(\sqrt{\frac{16\pi}{p}} \frac{\phi}{m_{\rm Pl}}) \quad \text{Exact solutions but ...}$ No natural end to inflation. 'Natural' inflation $V(\phi) = V_0[1+\cos\frac{\phi}{f}]$ Intermediate inflation $V(\phi) \propto \phi^{-\beta} \qquad \text{Also no natural end.}$

Some of these actually do not satisfy the condition of a minimum in which inflation ends; they permit inflation to continue forever. However, we shall see power-law inflation arising in a more satisfactory context shortly.

6.6.2 Multi-field theories

A recent trend in inflationary model building has been the exploration of models with more than one scalar field. The classic example is the hybrid inflation model [15], which seems particularly promising for particle physics model building. It has a potential with two fields ϕ and ψ of the form

$$V(\phi, \psi) = \frac{\lambda}{4} \left(\psi^2 - M^2 \right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \lambda' \phi^2 \psi^2.$$
 (48)

which is illustrated in Figure 5. When ϕ^2 is large, the minimum of the potential is at $\psi = 0$. The field rolls down this 'channel' until it reaches $\phi_{\rm inst}^2 = \lambda M^2/\lambda'$, at which point $\psi = 0$ becomes unstable and the field rolls into one of the true minima at $\phi = 0$ and $\psi = \pm M$.

While in the 'channel', which is where all the interesting behaviour takes place, this is just like a single field model with an effective potential for ϕ of the form

$$V_{\text{eff}}(\phi) = \frac{\lambda}{4}M^4 + \frac{1}{2}m^2\phi^2.$$
 (49)

This is a fairly standard form, the unusual thing being the constant term, which would not normally be allowed as it would give a present-day cosmological constant. The most interesting regime is where that constant dominates, and it gives quite an unusual phenomenology. In particular, the energy density during inflation can be much lower than normal while still giving suitably large density perturbations, and secondly the field ϕ can be rolling extremely slowly which is of benefit to particle physics model building.

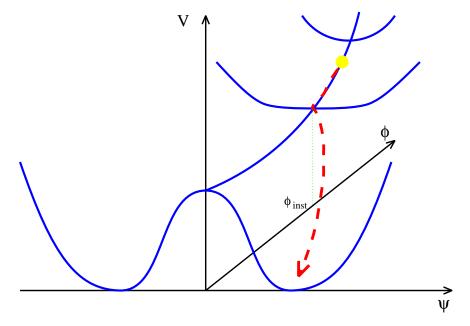


Figure 5: The potential for the hybrid inflation model. The field rolls down the channel at $\psi = 0$ until it reaches the critical ϕ value, then falls off the side to the true minimum at $\phi = 0$ and $\psi = \pm M$.

Within the more general class of two and multi-field inflation models, it is quite common for only one field to be dynamically important, as in the hybrid inflation model — this effectively reduces the situation back to the single field case of the previous subsection. However, it may also be possible to have more than one important dynamical degree of freedom. In that case there is no attractor behaviour giving a unique route into the potential minimum, as in the single field case; for example, if the potential is of the form of an asymmetric bowl one could roll into the base down any direction. In that situation, the model loses its predictive power, because the late-time behaviour is not independent of the initial conditions.⁶

6.6.3 Beyond general relativity

Rather than introduce an explicit scalar field to drive inflation, some theories modify the gravitational sector of the theory into something more complicated than general relativity [17]. Examples are

- Higher derivative gravity $(R + R^2 + \cdots)$.
- Jordan–Brans–Dicke theory.
- Scalar–tensor gravity.

The last two are theories where the gravitational constant may vary (indeed Jordan–Brans–Dicke theory is a special case of scalar–tensor gravity).

However, a clever trick, known as the *conformal transformation* [16], allows such theories to be rewritten as general relativity plus one or more scalar fields with some potential. Often,

⁶Of course, there is no requirement that the 'true' physical theory does have predictive power, but it would be unfortunate for us if it does not.

only one of those fields is dynamical which returns us once more to the original chaotic inflation scenario!

The most famous example is extended inflation [18]. In its original form, it transforms precisely into the power-law inflation model that we've already discussed, with the added bonus that it includes a proper method of ending inflation. Unfortunately though, this model is now ruled out by observations [8]! Indeed, models of inflation based on altering gravity are much more constrained than other types, since we know a lot about gravity and how well general relativity works [17], and many models of this kind are very vulnerable to observations.

6.6.4 Open inflation

Recently, some interest has been given to an idea with a very long history indeed, which is a way of generating an open universe from inflation.⁷ Often in the past it has been declared that this is either impossible or contrived; however, it can be readily achieved in models with quantum tunnelling from a false vacuum (a metastable state) followed by a second inflationary stage [19]. The tunnelling creates a bubble, and, incredibly, the region inside the expanding bubble looks just like an open universe, with the bubble wall corresponding to the initial (coordinate) singularity. These models are normally referred to as 'open inflation' or 'single-bubble' models.

These models are clearly very different from traditional inflation models, and may become the focus of extra attention should observational evidence for a low density universe continue to firm up. However I have no space to give them a proper discussion here and from now on will restrict discussion to the single-field chaotic inflation models.

6.7 Recap

The main points of this long Section are the following.

- Cosmological scalar fields, which were introduced long before inflation was thought of, provide a natural framework for inflation.
- Despite a wide range of motivations, most inflationary models are dynamically equivalent to general relativity plus a single scalar field with some potential $V(\phi)$.
- Within this framework, solutions describing inflation are easily found. Indeed, for many of the properties (amount of expansion, for example), we do not even need to solve the equations of motion.

With this information under our belts, we are now able to discuss the strongest motivation for the inflationary cosmology — that it is able to provide an explanation for the origin of structure in the universe.

7 Density Perturbations and Gravitational Waves

In modern terms, by far the most important property of inflationary cosmology is that it produces spectra of both density perturbations and gravitational waves. The density perturbations may be responsible for the formation and clustering of galaxies, as well as creating anisotropies in the microwave background radiation. The gravitational waves do not affect the

⁷That is, a genuinely open universe with hyperbolic geometry and no cosmological constant.

formation of galaxies, but as we shall see may contribute extra microwave anisotropies on the large angular scales sampled by the COBE satellite [2].

Studies of large-scale structure typically make some assumption about the initial form of these spectra. Usually gravitational waves are assumed not to be present, and the density perturbations to take on a simple form such as the scale-invariant Harrison–Zel'dovich spectrum, or a scale-free power-law spectrum. It is clearly highly desirable to have a theory which predicts the forms of the spectra. There are presently two rival models which do this, cosmological inflation and topological defects. I will return to the topic of topological defects towards the end of these lectures.

7.1Production during inflation

The ability of inflation to generate perturbations on large scales comes from the unusual behaviour of the Hubble length during inflation, namely that (by definition) the comoving Hubble length decreases. When we talk about large-scale structure, we are primarily interested in comoving scales, as to a first approximation everything is dragged along with the expansion. The qualitative behaviour of irregularities is governed by their scale in comparison to the characteristic scale of the Universe, the Hubble length.

In the big bang universe the comoving Hubble length is always increasing, and so all scales are initially much larger than it, and hence unable to be affected by causal physics. Once they become smaller than the Hubble length, they remain so for all time. In the standard scenarios, COBE sees perturbations on large scales at a time when they were much bigger than the Hubble length, and hence no mechanism could have created them.

Inflation reverses this behaviour, as seen in Figure 6. Now a given comoving scale has a more complicated history. Early on in inflation, the scale would be well inside the Hubble length, and hence causal physics can act, both to generate homogeneity to solve the horizon problem and to superimpose small perturbations. Some time before inflation ends, the scale crosses outside the Hubble radius (indicated by a circle in the lower panel of Figure 6) and causal physics becomes ineffective. Any perturbations generated become imprinted, or, in the usual terminology, 'frozen in'. Long after inflation is over, the scales cross inside the Hubble radius again. The perturbations we are interested in range from about the size of the present Hubble radius (i.e. the size of the presently observable universe) down to a few orders of magnitude less. On the scale of Figure 6, all interesting comoving scales lie extremely close together, and cross the Hubble radius during inflation very close together.

It's all very well to realize that the dynamics of inflation permits perturbations to be generated without violating causality, but we need a specific mechanism. That mechanism is quantum fluctuations. Inflation is trying as hard as it can to make the universe perfectly homogeneous, but it cannot defeat the Uncertainty Principle which ensures that there are always some irregularities left over. Through this limitation, it is possible for inflation to adequately solve the homogeneity problem and in addition leave enough irregularities behind to attempt to explain why the present universe is not completely homogeneous.

The size of the irregularities depends on the energy scale at which inflation takes place. It is outside the scope of these lectures to describe in detail how this calculation is performed (see e.g. Ref. [20] for such a description); I'll just briefly outline the necessary steps and then quote the result, which we can go on to apply.

- $\phi = \phi(t) + \delta\phi(\mathbf{x}, t)$ $\delta\phi = \sum (\delta\phi)_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}$ (a) Perturb the scalar field
- (b) Expand in comoving wavenumbers

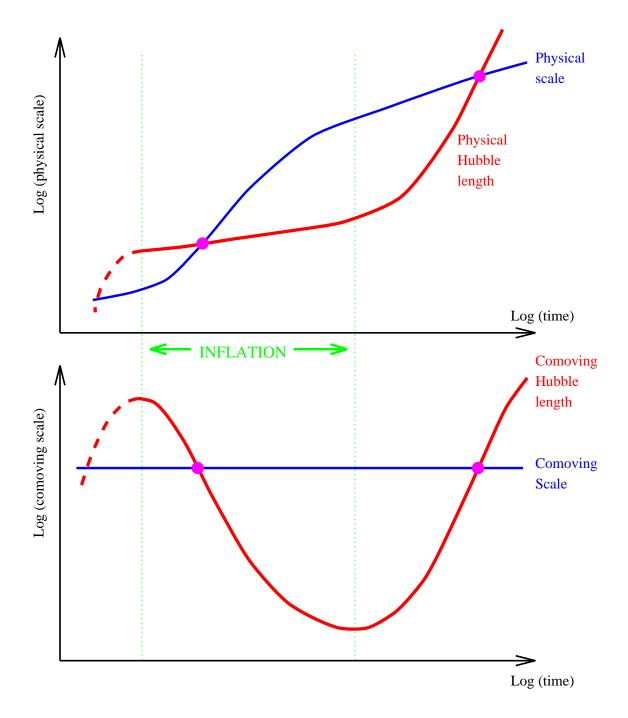


Figure 6: The behaviour of a given comoving scale relative to the Hubble length, both during and after inflation, shown using physical coordinates (upper panel) and comoving ones (lower panel).

 $\ddot{\delta\phi}_{\mathbf{k}} + 3H\dot{\delta\phi}_{\mathbf{k}} + [k^2/a^2 + V'']\delta\phi_{\mathbf{k}} = 0$ (c) Linearized equation for classical evolution

(d) Quantize theory

(e) Find solution with initial condition giving flat space quantum theory $(k \gg aH)$

 $\langle |\delta \phi_{\mathbf{k}}|^2 \rangle = H^2/2k^3$ $\mathcal{R} = H \delta \phi/\dot{\phi}$ (f) Find asymptotic value for $k \ll aH$

(g) Relate field perturbation to metric or curvature perturbation

Some important points are

- The details of this calculation are extremely similar to those used to calculate the Casimir effect (a quantum force between parallel plates), which has been tested in the laboratory.
- The calculation itself is not controversial, though some aspects of its interpretation (in particular concerning the quantum to classical transition) are.
- Exact analytic results are not known for general inflation models (though linear theory results for arbitrary models are easily calculated numerically). The results I'll be quoting will be lowest-order in the SRA, which is good enough for present observations.
- Results are known to second-order in slow-roll for arbitrary inflaton potentials. Powerlaw inflation is the only standard model for which exact results are known. In some other cases, high accuracy approximations give better results (e.g. small-angle approximation in natural or hybrid inflation).

The formulae for the amplitude of density perturbations, which I'll call $\delta_{\rm H}(k)$, and the gravitational waves, $A_{\rm G}(k)$, are⁸

$$\delta_{\rm H}(k) = \sqrt{\frac{512\pi}{75}} \frac{V^{3/2}}{m_{\rm Pl}^3 |V'|} \bigg|_{k=aH} , \qquad (50)$$

$$A_{\rm G}(k) = \sqrt{\frac{32}{75}} \frac{V^{1/2}}{m_{\rm Pl}^2} \bigg|_{k=aH} .$$
 (51)

Here k is the comoving wavenumber; the perturbations are normally analyzed via a Fourier expansion into comoving modes. The right-hand sides of the above equations are to be evaluated at the time when k = aH during inflation, which for a given k corresponds to some particular value of ϕ . We see that the amplitude of perturbations depends on the properties of the inflaton potential at the time the scale crossed the Hubble radius during inflation. The relevant number of e-foldings from the end of inflation is given by [7]

$$N \simeq 62 - \ln \frac{k}{a_0 H_0} + \text{numerical correction},$$
 (52)

where 'numerical correction' is a typically smallish (order one or a few) number which depends on the energy scale of inflation, the duration of reheating and so on. Normally it is a perfectly fine approximation to say that the scales of interest to us crossed outside the Hubble radius 60 e-foldings before the end of inflation. Then the e-foldings formula

$$N \simeq -\frac{8\pi}{m_{\rm Pl}^2} \int_{\phi}^{\phi_{\rm end}} \frac{V}{V'} d\phi , \qquad (53)$$

⁸The precise normalization of the spectra is arbitrary, as are the number of powers of k included. I've made my favourite choice here (following [7, 20]), but whatever convention is used the normalization factor will disappear in any physical answer. For reference, the usual power spectrum P(k) is proportional to $k\delta_{\rm H}^{2}(k)$.

tells us the value of ϕ to be substituted into Eqs. (50) and (51).

7.2 A worked example

The easiest way to see what is going on is to work through a specific example, the $m^2\phi^2/2$ potential which we already saw in Section 6.3. We'll see that we don't even have to solve the evolution equations to get our predictions.

- 1. Inflation ends when $\epsilon = 1$, so $\phi_{\rm end} \simeq m_{\rm Pl}/\sqrt{4\pi}$.
- 2. We're interested in 60 e-foldings before this, which from Eq. (46) gives $\phi_{60} \simeq 3m_{\rm Pl}$.
- 3. Substitute this in:

$$\delta_{\rm H} \simeq 12 \, \frac{m}{m_{\rm Pl}} \quad ; \quad A_{\rm G} \simeq 1.4 \, \frac{m}{m_{\rm Pl}}$$

4. Reproducing the COBE result requires $\delta_{\rm H} \simeq 2 \times 10^{-5}$ [21] (provided $A_{\rm G} \ll \delta_{\rm H}$), so we need $m \simeq 10^{-6} m_{\rm Pl}$.

7.3 Observational consequences

Observations have moved on beyond us wanting to know the overall normalization of the potential. The interesting things are

- 1. The scale-dependence of the spectra.
- 2. The relative influence of the two spectra.

These can be neatly summarized using the slow-roll parameters ϵ and η we defined earlier [8]. The standard approximation used to describe the spectra is the **power-law approxima-**

tion, where we take

$$\delta_{\rm H}^2(k) \propto k^{n-1} \; ; \; A_{\rm G}^2(k) \propto k^{n_{\rm G}} \,,$$
 (54)

where the spectral indices n and $n_{\rm G}$ are given by

$$n - 1 = \frac{d \ln \delta_{\rm H}^2}{d \ln k}$$
 ; $n_{\rm G} = \frac{d \ln A_{\rm G}^2}{d \ln k}$. (55)

The power-law approximation is usually valid because only a limited range of scales are observable, with the range 1 Mpc to 10^4 Mpc corresponding to $\Delta \ln k \simeq 9$.

The crucial equation we need is that relating ϕ values to when a scale k crosses the Hubble radius, which from Eq. (53) is

$$\frac{d\ln k}{d\phi} = \frac{8\pi}{m_{\rm Pl}^2} \frac{V}{V'}.\tag{56}$$

(since within the slow-roll approximation $k \simeq \exp N$). Direct differentiation then yields [8]

$$n = 1 - 6\epsilon + 2\eta, \tag{57}$$

$$n_{\rm G} = -2\epsilon \,, \tag{58}$$

where now ϵ and η are to be evaluated on the appropriate part of the potential.

MODEL	POTENTIAL	n	R	Note
Polynomial	ϕ^2	0.97	0.1	
chaotic inflation	ϕ^4	0.95	0.2	
Power-law inflation	$\exp(-\lambda\phi)$	any $n < 1$	$2\pi(1-n)$	Inc. extended inflation
'Natural' inflation	$1 + \cos(\phi/f)$	any $n < 1$	0	Basis for 'tilted' models
Hybrid inflation (standard)	$1 + B\phi^2$	1	0	Gives 'simplest' spectra
Hybrid inflation (extreme)	$1 + B\phi^2$	1 < n < 1.15	~ 0	'Blue' spectra

Table 1: The spectral index and gravitational wave contribution for a range of inflation models.

Finally, we need a measure of the relevant importance of density perturbations and gravitational waves. The natural place to look is the microwave background; a detailed calculation which I cannot reproduce here (see e.g. Ref. [7]) gives

$$R \equiv \frac{C_{\ell}^{\text{GW}}}{C_{\ell}^{\text{DP}}} \simeq 4\pi\epsilon. \tag{59}$$

Here the C_{ℓ} are the contributions to the microwave multipoles, in the usual notation.⁹ From these expressions we immediately see

- If and only if $\epsilon \ll 1$ and $|\eta| \ll 1$ do we get $n \simeq 1$ and $R \simeq 0$.
- Because the coefficient in Eq. (59) is so large, gravitational waves can have a significant effect even if ϵ is quite a bit smaller than one.

Table 1 shows the predictions for a range of inflation models. Even the simplest inflation models can affect the large-scale structure modelling at a level comparable to the present observational accuracy. The predictions of the different models will be wildly different as far as future high accuracy observations are concerned.

Observations have some way to go before the power-law approximation becomes inadequate. Consequently ...

- Slow-roll inflation adds two, and only two, new parameters to large-scale structure.
- Although ϵ and η are the fundamental parameters, it is best to take them as n and R.
- Inflation models predict a wide range of values for these. Hence inflation makes no definite prediction for large-scale structure.
- However, this means that large-scale structure observations, and especially microwave background observations, can strongly discriminate between inflationary models. When they are made, most existing inflation models will be ruled out.

7.4 Tests of inflation

The moral of the previous Section was that different inflation models lead to very different models of structure formation, spanning a wide range of possibilities. That means, for example, that a definite measure of say the spectral index n would rule out most inflation models. But it would always be possible to find models which did give that value of n. Is there any way to try and test the idea of inflation, independently of the model chosen?

⁹Namely, $\Delta T/T = \sum a_{\ell m} Y_m^{\ell}(\theta, \phi), C_{\ell} = \langle |a_{\ell m}|^2 \rangle.$

The answer, in principle, is yes. In the previous Section we introduced three observables (in addition to the overall normalization), namely n, R and $n_{\rm G}$. However, they depend only on two fundamental parameters, namely ϵ and η [8]. We can therefore eliminate ϵ and η to obtain a relation between observables, the *consistency equation*

$$R = -2\pi n_{\rm G}. \tag{60}$$

This relation has been much discussed in the literature [22, 20]. It is *independent* of the choice of inflationary model (though it does rely on the slow-roll and power-law approximations).

The idea of a consistency equation is in fact very general. The point is that we have obtained two continuous functions, $\delta_{\rm H}(k)$ and $A_{\rm G}(k)$, from a single continuous function $V(\phi)$. This can only be possible if the functions $\delta_{\rm H}(k)$ and $A_{\rm G}(k)$ are related, and the equation quoted above is the simplest manifestation of such a relation.

Vindication of the consistency equation would be a remarkably convincing test of the inflationary paradigm, as it would be highly unlikely that any other production mechanism could entangle the two spectra in the way inflation does. Unfortunately though, measuring $n_{\rm G}$ is a much more challenging observational task than measuring n or R and may be beyond even next generation observations. Indeed, this is a good point to remind the reader that even if inflation is right, only one model can be right and it is perfectly possible (and maybe even probable, see Ref. [23]) that that model has a very low amplitude of gravitational waves and that they will never be detected.

8 Further Early Universe Topics

Most of my time has been spent discussing cosmological inflation and its consequences. It's time now to move on to a brief discussion of some of the other topics which fall under the umbrella of Early Universe Cosmology. Most of them are discussed at a high level of detail in the book by Kolb & Turner [3].

8.1 Baryogenesis

There is considerable observational evidence that the Universe is, by a vast majority, comprised of baryons rather than anti-baryons. Evidence within our solar system comes from the lack of annihilations experienced by any lunar or interplanetary probes, while from beyond the absence of antinucleons in cosmic rays, and of gamma rays produced in annihilations. The existence of horizons in the early Universe precludes the possibility of large-scale segregations needed to preserve a baryon-symmetric Universe.

In comparison to the number density of photons, the present number density of baryons is small indeed — about one baryon per ten billion photons. However, if one tracks this asymmetry back to early times, assuming baryon number conservation, then at early times one expects the baryons to be in thermal equilibrium with the photons and hence with the same number density. At that time, the fractional baryon asymmetry would be very small indeed; for every ten billion photons there would be ten billion anti-baryons and ten billion and one baryons. Once the Universe cools enough, the ten billion anti-baryons annihilate with the baryons to leave the small excess.

Reminiscent of say the flatness problem, we could imagine that baryon number is perfectly conserved in the Universe, and that the excess of baryons over anti-baryons is simply a feature of the initial conditions. But it would be enormously preferable to have a physical theory capable

of explaining the excess. The necessary ingredients were identified long ago by Sakharov [24], and are

Baryon number violation: Obviously necessary.

 ${\bf C}$ and ${\bf CP}$ violation: ${\bf C}$ is the charge conjugation operator and ${\bf P}$ the parity operator. Their violation is necessary in order select a preference for baryons or anti-baryons. Their violation is already observed in nature in K meson interactions (which do not however violate baryon number).

Non-equilibrium conditions: In equilibrium, reactions occur forwards and backwards with the same rate, so even if reactions violate baryon number, the forward and backward reactions will cancel out.

The original models for baryogenesis were based on Grand Unified Theories for particle interactions, which permit baryons to decay into leptons violating baryon number conservation. Non-equilibrium conditions arise naturally in an expanding Universe, from the changing relationship between the expansion rate and the key particle interaction rates. The standard scenario of this kind involves massive particles with baryon number violating decays. If their decay is slower than the expansion rate, they are unable to stay in equilibrium as the Universe expands, and go through a phase of being overabundant before decaying to generate a net baryon number. Because the Universe is much cooler by the time they decay, the reverse reaction is heavily suppressed. Much work was carried out on this type of scenario in the eighties (see Kolb & Turner [3] for a review).

In recent years, attention has been focussed in a different direction, electro-weak baryogenesis (for a review, see Ref. [25]). It was recognized that the electro-weak theory, while preserving baryon number in perturbative interactions, could non-perturbatively violate baryon number. A configuration doing this has become known as the sphaleron. Although at zero temperature this is a tiny effect, it was argued by Kuzmin et al. [26] that at high temperatures the suppression vanishes. This implies very rapid baryon number violation in the early Universe, even if we restrict ourselves to Standard Model interactions alone. In fact, strictly speaking it is the sum of baryon and lepton numbers which is violated; the difference between them, **B**-**L**, is conserved even by non-perturbative interactions.

Sphalerons have a drastic effect on any pre-existing **B**+**L** symmetry — they erase it.¹⁰ Consequently, it appears futile to try and make an asymmetry of this type at the GUT era, as it will later be destroyed. The simplest GUT, SU(5) [which is in any case ruled out by the lack of observed proton decays], can only create an asymmetry of this type. However, more complicated GUTs can violate **B**-**L**, so that when **B**+**L** is driven to zero a residual baryon asymmetry is left. Indeed, one interesting proposal is leptogenesis, where a lepton asymmetry is generated at high temperatures and the sphalerons used to convert part of it into a baryon asymmetry.

Although GUT baryogenesis is clearly then still possible, it would be nice to try and capitalize on the baryon number violating property of the electro-weak theory to create the baryon asymmetry within the standard model. Many proposals have been made to try and realize this, with a belief developing that it can only be viable if the electro-weak phase transition is quite strongly first-order, so as to maximize departures from equilibrium. No compelling model has yet been constructed, with the asymmetry coming out usually being too small. However, it is

¹⁰Note that there are effectively no observational constraints on the lepton number of the Universe, since the neutrino background cannot be directly observed.

tempting to believe that since the answers coming out of these calculations are not hopelessly wrong, there must be good chances that these models will be shown, after all, to be viable.

8.2 Topological defects

Topological defects — domain walls, cosmic strings, monopoles and textures — offer a rival theory to inflation for the origin of structure (for a full account of topological defects, see Vilenkin and Shellard [27]). They are irregularities formed in the Universe during phase transitions, and can occur when a scalar field has more than one minimum of its potential.

The simplest example is to consider a real scalar field ϕ with potential

$$V_0(\phi) = \lambda (\phi^2 - M^2)^2 \tag{61}$$

where λ and M are constants. This potential has two minimum, at $\phi = \pm M$, and possesses a reflection symmetry $\phi \leftrightarrow \phi$.

At high temperatures, this potential is dominated by temperature corrections; I'll assume they take a very simple form giving the effective potential as

$$V_T(\phi) = V_0(\phi) + \frac{1}{2}T^2\phi^2$$
(62)

At high temperatures, the temperature correction dominates and the minimum of the potential is at the origin; this is said to be the symmetric phase. However, the Universe cools as it expands and eventually $\phi = 0$ stops being the minimum. At low temperature, the field wishes to sit in one of the minima, at $\phi = \pm M$.

The lowest energy state of the system is for all the energy to reside in one of the minima, everywhere in the Universe. The trouble is, the field has to decide in which way to fall, and the existence of horizons means that the field in one region of the Universe cannot 'signal' to other, causally disconnected, regions which direction they are supposed to fall. Consequently, in widely separated regions the field makes independent choices, and is as likely to fall one way as the other.

The question is, what happens on the boundaries between those regions? The field must be continuous, so on the boundary it must smoothly evolve from $\phi = -M$ to $\phi = +M$. In doing so it must pass through $\phi = 0$, which is a region of high potential energy. Since any line drawn from the first point to the second must pass through $\phi = 0$, this potential energy must take the form of a sheet, known as a **domain wall**.

Domain walls are the simplest type of defect that can form, existing where, as in the example above, the minimum energy states (known as the **vacuum manifold**) are disconnected. More complicated types of vacuum state lead to other types of defect; take for example a complex scalar field

$$V = \lambda \left(\phi^{\dagger} \phi - M^2 \right)^2 \tag{63}$$

Here the minima are at $\phi = Me^{i\alpha}$, where $\alpha \in [0, 2\pi)$. This vacuum manifold is a circle. There are no domain wall solutions (since the vacuum is connected), but it is possible to form a defect known as a **cosmic string**.

Cosmic strings form when a loop drawn in space corresponds to a winding around the vacuum manifold. The different locations on the loop correspond to different angles α . Imagine contracting the loop to a point; continuity can only occur if the angles become degenerate, which can only happen if $\phi = 0$, i.e. if there is a location within the loop where the field is not in the vacuum state. Such locations form a line defect.

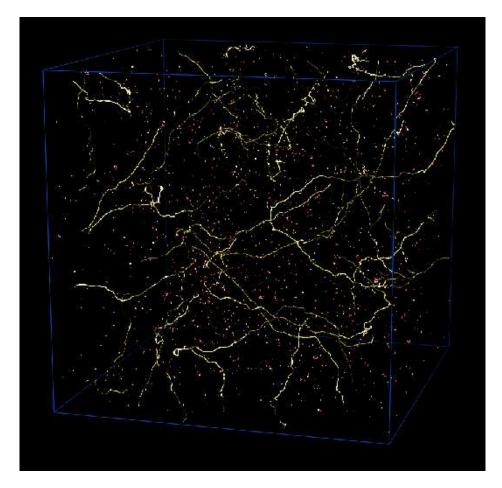


Figure 7: A string network can be very complicated; this simulation by Allen and Shellard [28] shows the evolution in the matter era. As well as long strings, there are numerous small loops of string.

More complicated vacuum manifolds lead to more complicated defects; if the vacuum is a sphere, then there are no domain walls or cosmic strings, but instead a point-like defect known as a magnetic monopole, ¹¹ which we already encountered as a motivation for inflation. An even more exotic type of defect is a texture, corresponding to a yet-more-complicated vacuum manifold.

The main cosmological interest in defects is that they are by their very nature inhomogeneous. That is, they are able to take a perfectly homogeneous Universe before the phase transition and insert inhomogeneities. If the defects are massive enough, then they may be seeds for structure formation; it turns out that if the symmetry breaking scale is that of Grand Unified Theories, then the energy density may be just about right. They therefore provide an alternative theory to inflation for the origin of cosmic structure.

Unfortunately, the defect theory for structure formation is much more complicated than the inflationary paradigm. The reason is because the defect dynamics are non-gravitational and non-linear. This means that the theoretical status of the subject is some way behind the

¹¹Strictly speaking, the theories I've been writing down have what is known as a global symmetry, and the defects should be known as global strings and monopoles. True cosmic strings and magnetic monopoles occur only when gauge fields are included, but the picture of their formation remains as I have described here.

observational situation, whereas inflation is some way ahead. Defects have therefore not yet been subjected to the most rigorous tests possible, and indeed may never be. Therefore, by far the majority of large-scale structure research has been based on the inflationary paradigm, at least implicitly through the assumption of gaussian density perturbations evolving only under gravity. I imagine that everything else in this conference will fall under the inflationary paradigm.

8.3 Dark matter

During this School we'll hear a considerable amount of motivation for the idea that the bulk of the matter in the Universe is in some as-yet-unknown form, known as *dark matter*. I won't attempt to review this evidence here; instead I'll concentrate on the particle physics aspects.

There are two commonly considered types of dark matter; either the dark matter is in the form of elementary particles, or it is in some form of compact object. Black holes would appear the most likely candidate of the latter type (low mass stars may provide some dark matter but nucleosynthesis constraints prevent them from being a candidate for all the dark matter), though there is no compelling theory suggesting that they can form in sufficient numbers.

Most attention is focussed on the possibility of elementary particles as dark matter. There are a range of possibilities, and many experiments are now active in the search for particle dark matter.

8.3.1 Neutrinos

Within the context of the standard model, there is only one type of particle which might possess a significant cosmological density, and that is the neutrino. Provided neutrinos are sufficiently light, which turns out to mean less than an MeV or so, they are strongly relativistic when they go out of thermal equilibrium. This implies that their number density is independent of mass (mass being irrelevant in the relativistic limit), and hence their energy density is proportional to their mass. A fairly simple calculation (see e.g. Kolb & Turner) gives their present abundance as

$$\Omega_{\nu} \simeq \frac{\sum_{i} m_{\nu_{i}}}{90h^{2} \,\text{eV}} \tag{64}$$

where the sum is over the neutrino families lighter than 1 MeV. We see that a neutrino mass around 30 eV (depending on the precise value of h) could explain the dark matter, and masses above this are excluded. This number is comparable to the experimental bound on the electron neutrino, and well below that of the muon or tau neutrino.

Such neutrinos would have been relativistic until fairly recently in the history of the Universe, and are known as **hot dark matter**. Evidence from structure formation suggests that having all the dark matter as hot dark matter will not lead to a satisfactory model, but it remains possible for a component of the dark matter to be. Neutrino oscillation experiments may be able to probe the relevant mass ranges.

If the neutrinos are more massive than 1 MeV, the relativistic freeze-out no longer applies and must be replaced with a non-relativistic one. In this regime, the present energy density begins to fall with increasing mass, and around a GeV or so we have another solution giving about a critical density in neutrinos. Such non-relativistic neutrinos are a form of **cold dark matter**.

8.3.2 WIMPS and axions

For further dark matter candidates, we must go beyond the Standard Model. The most discussed extension is supersymmetry, which at a stroke doubles the particle spectrum by associating a boson with every Standard Model fermion and a fermion with each Standard Model boson. These so-called superparticles possess a new quantum number, **R**-parity, which in the simplest models is conserved, guaranteeing that the lightest superparticle is stable. Its precise mass depends on the means by which supersymmetry is broken, but normally it is some or many GeV and it gives a cold dark matter candidate.

Another alternative is the axion, which arises as a solution to the strong-CP problem; that is, it provides a way of suppressing large CP violation in strong interactions. The phenomenology of the axion is complicated; although it has a very light mass, perhaps 10^{-5} eV, it is produced non-thermally and gives a cold dark matter candidate.

8.4 Primordial black holes

An interesting possibility is that black holes may form at some stage during the early Universe. This environment seems to be the only one in which black holes might be formed which are light enough that the process of Hawking evaporation might be important. Hawking discovered that black holes radiate with a temperature given by [29]

$$T_{\rm BH} = \frac{m_{\rm Pl}^2}{8\pi M_{\rm PH}} \tag{65}$$

This gives a lifetime τ of

$$\tau_{\rm BH} = \frac{M_{\rm BH}^3}{g_* m_{\rm Pl}^4} \tag{66}$$

where g_* is the number of particle degrees of freedom into which the black hole can decay. In more readily understood units, this is

$$\frac{\tau_{\rm BH}}{10^{10} \, \rm yrs} \simeq \left(\frac{M_{\rm BH}}{10^{15} \, \rm grams}\right)^3$$
 (67)

That is, a black hole with an initial mass of 10^{15} grams will have a lifetime equal to that of the present age of the Universe, and so will be evaporating at the present epoch. Much lighter holes will have evaporated long ago, while much heavier ones (such as those formed from stellar collapse) will have negligible evaporation.

There are at least three ways in which light primordial black holes can be formed during the early Universe.

- 1. Black holes may form from large density perturbations [30, 31], for example induced near the end of an inflationary epoch.
- 2. They may form in phase transitions, particularly strongly first-order transitions which proceed explosively by bubble nucleation [32]; some inflation models have inflation ending this way.
- 3. They may form through thermal fluctuations [33] during the very early Universe. This process is only efficient at extremely high temperatures, and any holes formed this way would be diluted away were there a subsequent period of inflation.

Primordial black holes are observationally interesting, because they redshift away as matter (i.e. as the third power of the scale factor), while the early Universe is normally assumed to be radiation dominated. If the black holes form early enough, it is therefore quite easy for them to come to dominate the energy density of the Universe even if their initial density at formation is a tiny fraction of the total. That enables one to place a variety of constraints on them.

- For black holes of masses above 10¹⁵ grams, the only constraint is that they must not contribute too much to the energy density of the Universe, i.e. they should not have more than a critical density. If they are at this density, then they are a cold dark matter candidate.
- Black holes evaporating today offer very powerful constraints, most importantly those from the γ ray background [34]. This limits black holes in this mass range to contribute at most orders of magnitude below the critical density.
- Black holes in the range 10⁹ grams to 10¹⁵ grams would have evaporated at earlier stages in the history of the Universe, and may have interfered with early processes such as nucleosynthesis. Again they are very strongly constrained.
- A more speculative possibility is that black holes don't evaporate away completely, but instead leave some stable relic. If so, then the constraints can strengthen considerably as these relics can contribute significantly to the present energy density, even if the initial holes were all so light as to have evaporated away [35].

A detailed summary of all of these can be found in Carr et al. [31].

9 Summary

This has been a brief introduction to a range of Early Universe topics, with only inflation covered in any depth at all. The main thrust I have been aiming to emphasize is that this area of research is becoming more and more a proper area of science, in the sense that models are being falsified and a large number of experiments promise to make decisive inroads into determining which, if any, of these ideas are on the right track. Progress is certain through our improved understanding of structure in the Universe, with microwave background satellite experiments poised to reveal the present state of the Universe with unprecedented accuracy. More speculatively, informative surprises may come our way from a number of sources, for example direct detection of dark matter or a convenient nearby supernova.

Acknowledgments

The author was supported by the Royal Society.

References

- [1] G. F. Smoot et al., Astrophys. J. **396**, L1 (1992).
- [2] C. L. Bennett et al., Astrophys. J. **464**, L1 (1996).

- [3] E. W. Kolb and M. S. Turner, *The Early Universe*, Addison-Wesley, Redwood City, California (1990) [updated paperback edition 1994].
- [4] A. D. Linde, *Particle Physics and Inflationary Cosmology*, Harwood Academic, Chur, Switzerland (1990).
- [5] T. Padmanabhan, Structure Formation in the Universe, Cambridge University Press (1993); P. J. E. Peebles, Principles of Physical Cosmology, Princeton University Press (1993); P. Coles and F. Lucchin, Cosmology: The Origin and Evolution of Cosmic Structure, Wiley, Chichester, Great Britain (1995).
- [6] J. V. Narlikar and T. Padmanabhan, Ann. Rev. Astron. Astrophys. 29, 325 (1991).
- [7] A. R. Liddle and D. H. Lyth, Phys. Rep **231**, 1 (1993).
- [8] A. R. Liddle and D. H. Lyth, Phys. Lett. B 291, 391 (1992).
- [9] D. S. Salopek and J. R. Bond, Phys. Rev. D 42, 3936 (1990).
- [10] A. R. Liddle, P. Parsons and J. D. Barrow, Phys. Rev. D 50, 7222 (1994).
- [11] L. Kofman, A. D. Linde and A. A. Starobinsky, Phys. Rev. Lett. 73, 3195 (1994); A. D. Linde, astro-ph/9601004; L. Kofman, astro-ph/9605155.
- [12] Y. Shtanov, J. Traschen and R. Brandenberger, Phys. Rev. D 51, 5438 (1995); D. Boyanovsky, M. D'Attanasio, H. de Vega, R. Holman, D.-S. Lee and A. Singh, Phys. Rev. D 52, 6805 (1995).
- [13] MAP home page at http://map.gsfc.nasa.gov/(1996).
- [14] COBRAS/SAMBA home page at http://astro.estec.esa.nl/SA-general/Projects/Cobras/cobras.html (1996).
- [15] A. D. Linde, Phys. Lett. B 259, 38 (1991), Phys. Rev. D 49, 748 (1994); E. J. Copeland,
 A. R. Liddle, D. H. Lyth, E. D. Stewart and D. Wands, Phys. Rev. D 49, 6410 (1994).
- [16] B. Whitt, Phys. Lett. 145B, 176 (1984); K. Maeda, Phys. Rev. D 39, 3159 (1989); D. Wands, Class. Quant. Grav. 11, 269 (1994).
- [17] C. M. Will, Theory and Experiment in Gravitational Physics, Cambridge University Press (1993).
- [18] D. La and P. J. Steinhardt, Phys. Rev. Lett. 62, 376 (1989); E. W. Kolb, Physica Scripta T36, 199 (1991).
- [19] J. R. Gott, Nature 295, 304 (1982); M. Sasaki, T. Tanaka, K. Yamamoto and J. Yokoyama, Phys. Lett. B 317, 510 (1993); M. Bucher, A. S. Goldhaber and N. Turok, Phys. Rev. D 52, 3314; A. D. Linde and A. Mezhlumian, Phys. Rev. D 52, 6789 (1995).
- [20] J. E. Lidsey, A. R. Liddle, E. W. Kolb, E. J. Copeland, T. Barriero and M. Abney, to appear, Rev. Mod. Phys., astro-ph/9508078.
- [21] E. F. Bunn, A. R. Liddle and M. White, Phys. Rev. D 54, 5917R (1996).

- [22] E. J. Copeland, E. W. Kolb, A. R. Liddle and J. E. Lidsey, Phys. Rev. D 48, 2529 (1993), 49, 1840 (1994).
- [23] D. H. Lyth, hep-ph/9606387.
- [24] A. D. Sakharov, JETP Letters 5, 24 (1967).
- [25] A. D. Dolgov, Phys. Rep. **222**, 309 (1992).
- [26] V. A. Kuzmin, V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B155, 36 (1985).
- [27] A. Vilenkin and E. P. S. Shellard, Cosmic Strings and Other Topological Defects, Cambridge University Press (1994).
- [28] B. Allen and E. P. S. Shellard, Phys. Rev. Lett. **64**, 119 (1990).
- [29] S. W. Hawking, Nature **248**, 30 (1974).
- [30] B. J. Carr, Astrophys. J. **201**, 1 (1975).
- [31] B. J. Carr, J. H. Gilbert and J. E. Lidsey, Phys. Rev. D 50, 4853 (1994).
- [32] J. D. Barrow, E. J. Copeland, E. W. Kolb and A. R. Liddle, Phys. Rev. D 43, 984 (1991).
- [33] D. J. Gross, M. J. Perry and L. G. Yaffe, Phys. Rev. D 25, 330 (1982); J. I. Kapusta, Phys. Rev. D 30, 831 (1984).
- [34] J. H. MacGibbon and B. J. Carr, Astrophys. J. **371**, 447 (1991).
- [35] J. D. Barrow, E. J. Copeland and A. R. Liddle, Phys. Rev. D 46, 645 (1992).

Constants and Conversion Factors

ſ	Newton's constant	G	$6.672 \times 10^{-11} \mathrm{m}^3 \mathrm{kg}^{-1} \mathrm{sec}^{-2}$		
	Speed of light	c	$2.998 \times 10^8 \mathrm{msec^{-1}}$	or	$3.076 \times 10^{-7} \mathrm{Mpc} \mathrm{yr}^{-1}$
	Reduced Planck constant	$\hbar = h/2\pi$	$1.055 \times 10^{-34} \mathrm{m}^2\mathrm{kg}\mathrm{sec}^{-1}$		
	Boltzmann constant	$k_{ m B}$	$1.381 \times 10^{-23} \mathrm{JK^{-1}}$	or	$8.619 \times 10^{-5} \mathrm{eV} \mathrm{K}^{-1}$
	Radiation constant	α	$7.565 \times 10^{-16} \mathrm{J}\mathrm{m}^{-3}\mathrm{K}^{-4}$		
	Planck mass	$m_{ m Pl}$	$2.179 \times 10^{-8} \mathrm{kg}$	or	$1.22\times 10^{19}\mathrm{GeV}$

Table 2: Some fundamental constants.

```
1 \, \mathrm{pc} = 3.262 \, \mathrm{light \; years} = 3.086 \times 10^{16} \, \mathrm{m}
1 \, \mathrm{yr} = 3.16 \times 10^{7} \, \mathrm{sec}
1 \, \mathrm{eV} = 1.602 \times 10^{-19} \, \mathrm{J}
1 \, M_{\odot} = 1.989 \times 10^{30} \, \mathrm{kg}
1 \, \mathrm{J} = 1 \, \mathrm{kg} \, \mathrm{m}^{2} \, \mathrm{sec}^{-2}
```

Table 3: Some conversion factors.